

UNKNOTTING NUMBER FOR LORENZ KNOTS

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ABSTRACT. The unknotting number of a positive braid with n strands and k intersections is known to be equal to $(k - n + 1)/2$. We consider Lorenz knots (which are positive braids) and, using a different method, find their unknotting numbers in terms of their positions on the Lorenz attractor.

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1. INTRODUCTION

In this paper we calculate unknotting numbers for some knots arising in dynamical systems.

Given a flow ϕ_t on a 3-manifold having a hyperbolic structure on its chain recurrent set, the link of periodic orbits of ϕ_t is in bijective correspondence with the link of periodic orbits (up to at most two exceptional orbits) on a particular branched 2-dimensional manifold called template or the knot-holder ([BW2], Theorem 2.1).

By a template (a knot-holder) H is meant a branched 2-manifold $H \subset M^3$, $H \neq \emptyset$, together with a semi-flow $\bar{\phi}_t$ on H such that H has an atlas consisting of 2 types of charts, a joining chart and a splitting chart, as on Figure 1. Each



Figure 1.

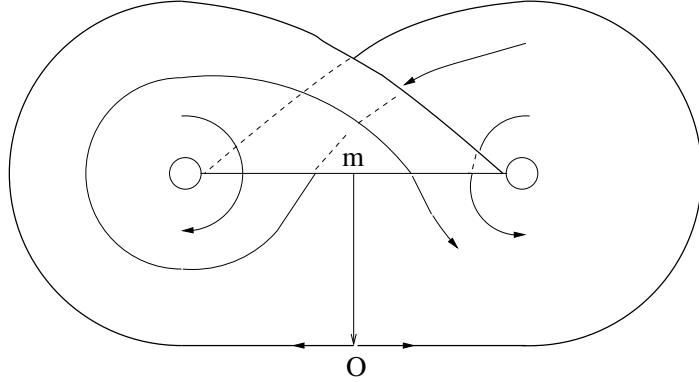


Figure 2.

component of the intersection of two charts is the bottom of one and a top of the other. The flow $\bar{\phi}_t$ is shown on the Figure 1. From the set θ the flow goes outwards.

We chose the simplest template, the Lorenz attractor, see Figure 2, corresponding to the flow in S^3 generated by the Lorenz equations ([BW1]). This template has an atlas consisting of two charts: a joining chart and a splitting chart. Knots living on this template are called Lorenz knots. All Lorenz knots are positive braids ([BW1]), and it was proven by Charles Livingston in [L], Corollary 7, that τ -invariant of a positive braid is equal to $(k - n + 1)/2$, where k is the number of intersections, and n is the number of strands of the braid. This gives a lower bound for the unknotting number, and since it is also an upper bound ([BoW], §4), the unknotting number of a positive braid is equal to $(k - n + 1)/2$.

In this paper we calculate, using a different method, the unknotting numbers of Lorenz knots in terms of their positions on Lorenz attractor. Here is an outline of the paper. In section 2 we use symbolic dynamics and introduce an additional geometric description of Lorenz knots to formulate the main result of this paper, Theorem 2.3. In section 3 we give necessary background in grid homology and τ -invariant. Sections 4-7 contain the proof of Theorem 2.3.

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2. LORENZ KNOTS

Let H be the Lorenz attractor, see Figure 2. Denote by I the branch set of H . The positive orbit through m , $\bar{\phi}_t(m)$, $t > 0$ approaches O as $t \rightarrow \infty$. For $x \in I - \{m\}$ there is a first return (Poincaré) image, $f(x) \in I$. The orbit from x to $f(x)$ goes around the left hole and in front for $x < m$, around the right hole and in back for $x > m$. Since the orbits don't intersect, $f(x)$ is monotonically increasing on both sub-intervals : $I_1 = \{x \in I, x < m\}$ and $I_2 = \{x \in I, x > m\}$.

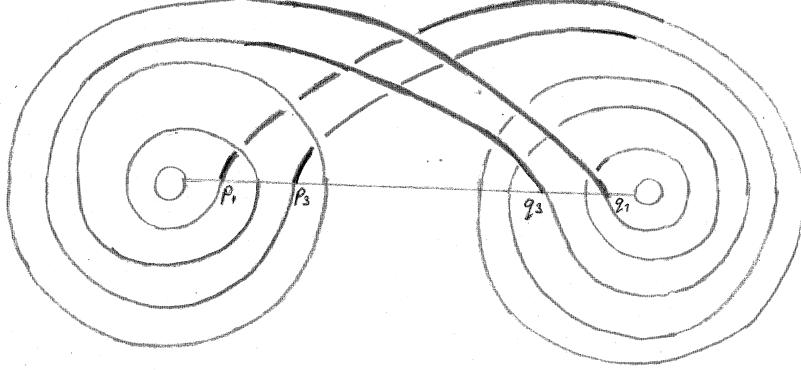


Figure 3.

Consider symbolic dynamics on I generated by the first return map f . To each point z we assign a finite or infinite sequence $k_0(z), k_1(z), \dots$ where

$$k_0(z) = \begin{cases} x & \text{if } z \text{ is to the left of } m \\ 0 & \text{if } z = m \\ y & \text{if } z \text{ is to the right of } m; \end{cases}$$

and $k_i(z)$ is defined iff $f^i(z)$ is defined by $k_i(z) = k_0(f^i(z))$. The sequences k are lexicographically ordered by setting $x < 0 < y$.

Proposition 2.1. ([BW1], Proposition 2.4.1)

The map $z \rightarrow k(z)$ is a 1-to-1 order preserving correspondence between the points of the branch set and the set of all sequences k_0, k_1, \dots such that

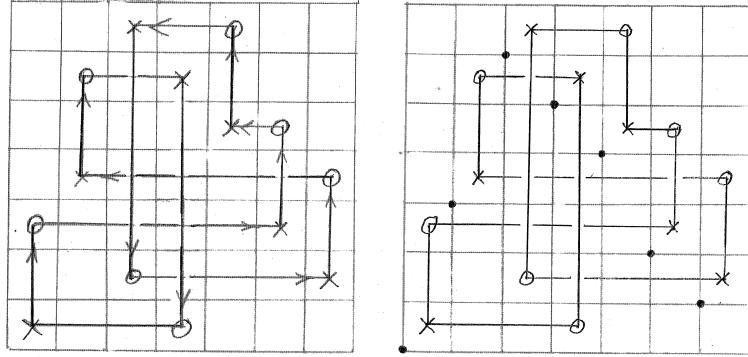
- (a) each $k_i = x, y$ or 0
- (b) the sequence terminates with k_i iff $k_i = 0$.

Theorem 2.2. ([BW1] Corollary 2.4)

The periodic orbits of $\bar{\phi}_t$ correspond 1-to-1 with the cyclic permutation classes of finite aperiodic words in x and y .

Example 2.1. On Figure 3 we have a diagram of the knot $K = x^3y^3xy^2$.

Given a Lorenz knot K corresponding to the word $x^{\alpha_1}y^{\beta_1}x^{\alpha_2} \dots x^{\alpha_t}y^{\beta_t}$, let $a = \sum_1^t \alpha_i$, $b = \sum_1^t \beta_i$, and t be the trip number: the number of strands going from the left to the right (equal to the number of strands going from the right to the left). To formulate our result we need to introduce some notations. We want to describe the positions of the strands going from the right to the left, μ_i , and from the left to the right, ν_i . To the knot K there corresponds a finite set of its intersection points with the branch set I . Denote them by $p_1 < p_2 \dots < p_a < q_b < \dots < q_1$; $p_i \in I_1$, $q_i \in I_2$. These points are permuted by the first return map f . We have: either $p_i = f(p_j)$ or $p_i = f(q_l)$. We are interested in the case when $p_i = f(q_l)$ for some l . Let $\{p_1, \dots, p_a\} \cap f(\{q_b, \dots, q_1\}) = \{p_{\mu_1} < p_{\mu_2} < \dots < p_{\mu_t}\}$. Similarly, let $\{q_b, \dots, q_1\} \cap f(\{p_1, \dots, p_a\}) = \{q_{\nu_t} < \dots < q_{\nu_1}\}$. This defines μ_i and ν_i uniquely. Note that $\mu_1 = 1$, since if $p_1 = f(p_j)$ for some j , by monotonicity of f , $f(p_1)$ must be to the left of p_1 which contradicts our numeration. So $p_1 = f(q_l)$ for some l . For the same reason $\nu_1 = 1$.



Figures 4 a,b .

Example 2.2. For the knot K on Figure 3 we have $\mu_1 = 1$, $\mu_2 = 3$, $\nu_1 = 1$, $\nu_2 = 3$.

We prove the following theorem:

Theorem 2.3. *The unknotting number of a Lorenz knot K is given by the formula*

$$(2.1) \quad u(K) = \frac{1}{2}((a+b)(t-1) - \sum_1^t \mu_i - \sum_1^t \nu_i + (t+1)).$$

The plan of the proof: we turn the diagram of a Lorenz knot into a grid diagram. We find a lower bound on unknotting number using the τ -invariant of the knot. We show that this number of crossing changes is enough to unknot the knot.

3. GRID DIAGRAM AND τ -INVARIANT OF A KNOT, BACKGROUND

There is a theory of grid homology for knots and links, introduced by Peter S. Ozsváth and Zoltan Szabó. In particular they define the τ -invariant of a knot from its grid homology and prove that it gives a lower bound for the unknotting number:

$$(3.1) \quad |\tau(K)| \leq u(K).$$

Since we will not use the definition of τ -invariant we will not present it here. Rather we will use a lower bound for τ -invariant obtained from a grid diagram of a knot.

Grid diagram is $n \times n$ grid on the plane with n small squares marked by X and another n small squares marked by O , so that there is exactly one X and one O in each column and in each row. In each column we connect X to O by an oriented segment, and in each row we connect O to X by an oriented segment. We arrange that vertical segments go over horizontal ones. Clearly, such a diagram defines an oriented link, and every oriented link in S^3 can be presented by a grid diagram. (See Figure 4a.)

We transfer our planar grid diagrams to the torus T obtained by identifying the top-most segment with the bottom-most one and the left-most segment with the right-most one. In the torus horizontal and vertical arcs of the grid become horizontal and vertical circles.

A grid state is an n -tuple of intersection points $\mathbf{x} = \{x_1, \dots, x_n\}$ of vertical and horizontal circles satisfying the requirement that each vertical circle contains exactly one of the elements of \mathbf{x} and each horizontal circle contains exactly one

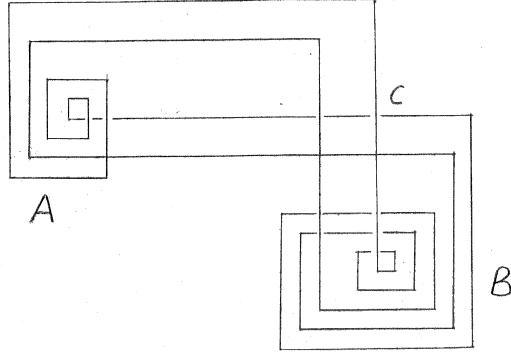


Figure 5.

element of \mathbf{x} . Let $S(G)$ denote the set of grid states of the grid diagram G . There is a bi-grading on $S(G)$: Maslov and Alexander gradings.

Definition 1. ([OSS], Definition 4.3.1) Let P, Q be two collections of finitely many points in the plane \mathbb{R}^2 . Let $I(P, Q)$ be the number of pairs $(p_1, p_2) \in P$ and $(q_1, q_2) \in Q$ with $p_1 < q_1, p_2 < q_2$. We symmetrize this, defining

$$J(P, Q) = \frac{I(P, Q) + I(Q, P)}{2}.$$

Then we extend J bi-linearly over formal sums of subsets of the plane.

We will need only the Alexander grading. It can be calculated by the formula:

$$(3.2) \quad A(\mathbf{x}) = - \sum_{x \in \mathbf{x}} w_K(x) + \frac{1}{2}(J(O, O) - J(X, X)) - \frac{(n-1)}{2},$$

where $w_K(x)$ is the winding number of the knot around the point x and n is the grid number. We will be interested in a particular grid state \mathbf{x}^- which occupies the lower left corner of each square marked with X . (See Figure 4b.) For this state it is proven ([OSS] Proposition 6.3.15) that

$$(3.3) \quad A(\mathbf{x}^-) \leq -\tau(K).$$

4. FROM LORENZ DIAGRAM TO GRID DIAGRAM

It is easy to turn the diagram of a knot on the Lorenz template into a grid diagram, see Figure 5.

The strands going from left to right become long vertical strands, and the strands going from right to left become long horizontal strands. We divide our diagram onto upper left part A , lower right part B and the central square C , where the vertical and horizontal strands intersect. Note that all intersections on the diagram happen on long vertical and long horizontal strands since short vertical and horizontal segments don't intersect each other (except for the end points). Here we used the following convention: on the grid diagram all vertical segments going from X to O fit into two groups: long vertical strands going from A to B and short vertical segments belonging either to A or to B . Similarly, all horizontal segments going from O to X are either long horizontal strands going from B to A , or short horizontal segments belonging to either B or A .

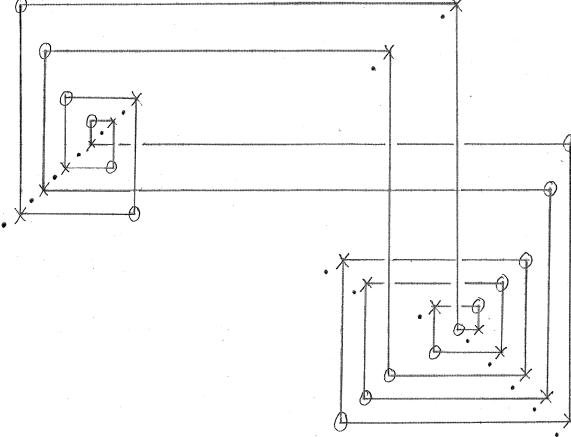


Figure 6.

Note that the grid number of the constructed diagram is $n = 2a + 2b - t$.

5. LOWER BOUND FOR UNKNOTTING NUMBER

Let \mathbf{x}^- be the grid state occupying lower left corners of squares marked with X , Figure 6.

We calculate its Alexander grading using the formula

$$(5.1) \quad A(\mathbf{x}^-) = - \sum_{x \in \mathbf{x}^-} w_K(x) + \frac{1}{2}(J(O, O) - J(X, X)) - \frac{(n-1)}{2},$$

Theorem 5.1.

$$(5.2) \quad A(\mathbf{x}^-) = \frac{1}{2}((a+b)(t-1) - \sum_1^t \mu_i - \sum_1^t \nu_i + (t+1)).$$

Proof. First we calculate $A'(\mathbf{x}^-) = - \sum_{x \in \mathbf{x}^-} w_K(x) = - \sum_A w_K(x) - \sum_B w_K(x)$, where \sum_A is the sum over upper $2a$ elements of \mathbf{x}^- and \sum_B is the sum over lower $2b - t$ elements of \mathbf{x}^- , see Figure 6. Since for $x \in A$, $-w_K(x)$ is equal to the total number of vertical segments to the left of it,

$$- \sum_A w_K(x) = 0 + 1 + 2 + \dots + (a-1) + a + (a-1) + \dots + 1 = a^2.$$

To calculate $-\sum_B w_K(x)$ note that for b rightmost points $x \in \mathbf{x}^-$, $-w_K(x)$ is equal to negative number of vertical segments to the right of x , so these x contribute $-(0 + 1 + \dots + b-1) = -\frac{b(b-1)}{2}$. Only $b-t$ points remain to the left of the first vertical strand. For them it is convenient to calculate $-w_K$ as negative number of vertical segments and vertical strands to the left of them. They contribute $0 + 1 + \dots + (b-t-1)$ plus vertical strand number k adds 1 to $\nu_k - 1 - (k-1)$ points (There are $\nu_k - 1$ vertical segments to the right of strand number ν_k till the first strand and $k-1$ of them are long strands. The number of points $x \in \mathbf{x}^-$ in the fragment of the diagram that we consider now is the number of short segments).

$$\begin{aligned}
-\sum_B w_K(x) &= -\frac{b(b-1)}{2} - \frac{(b-t-1)(b-t)}{2} - \sum_1^t ((\nu-1) - (k-1)) \\
&= -b^2 + b + bt - \sum_1^t \nu_k,
\end{aligned}$$

and

$$A'(\mathbf{x}^-) = -\sum_{x \in \mathbf{x}^-} w_K(x) = a^2 - b^2 + b + bt - \sum_1^t \nu_k.$$

Now $J(O, O)$ and $J(X, X)$ are calculated straightforwardly. There are $2b - O'$ strands in part $B \cup C$ that contribute $b(2b-1)$. In part $A \cup C$ each O on the k -th long horizontal strand that has position μ_k contributes the amount equal to the amount of short horizontal segments below this strand. There are $a - \mu_k$ horizontal segments below μ_k and $t - k$ of them are long horizontal strands. So there are $a - \mu_k - (t - k)$ short horizontal segments (O 's) below k -th strand. We get

$$J(O, O) = \sum_{k=1}^t (a - t - (\mu_k - k)) + b(2b-1) = at - \sum_1^t \mu_k - t^2 + \frac{t(t+1)}{2} + b(2b-1).$$

Similar argument gives

$$\begin{aligned}
J(X, X) &= \sum_{k=1}^t (b - t - (\nu_k - k)) + a(2a-1) \\
&= bt - \sum_1^k \nu_k - t^2 + \frac{t(t+1)}{2} + 2a^2 - a.
\end{aligned}$$

Plugging all these results into formula (5.1), we get the desired formula (5.2). \square

6. INTERSECTION POINTS OF A GRID DIAGRAM

Lemma 6.1. *The number of crossings on the grid diagram for K is*

$$(6.1) \quad c = (a+b)t - \sum \mu_k - \sum \nu_k + t$$

Proof. Note that all intersections happen on long vertical and long horizontal strands. Take a vertical strand and calculate the number of intersections on part B of it. That is the number of short horizontal segments that go down after intersecting the strand. It is equal to the total number of vertical segments to the left of our strand minus the number of the long vertical strands to the left of it. If the k -th strand occupies position ν_k , then the number of intersections on it is

$$c_k = (b - \nu_k) - (t - k)$$

Taking the sum over k we get

$$c_B = bt - \sum_1^t \nu_k - t^2 + \frac{t(t+1)}{2},$$

where c_B is the number of intersections in part B . Similarly, for horizontal strands

$$c_A = at - \sum_1^t \mu_k - t^2 + \frac{t(t+1)}{2},$$

where c_A is the number of intersections in part A . Then we add t^2 intersections of long vertical strands with long horizontal strands to get total number of intersections

$$c = (a + b)t - \sum_1^t \mu_k - \sum_1^t \nu_k + t$$

□

7. UNKNOTTING THE KNOT

Theorem 7.1. *Unknotting number of a Lorenz knot K , $u(K)$, satisfies the inequality*

$$(7.1) \quad u(K) \leq \frac{1}{2}((a + b)(t - 1) - \sum_1^t \mu_i - \sum_1^t \nu_i + (t + 1)).$$

Proof. It is a classical observation that if we trace a diagram of a knot going first time over and second time under at each intersection, then it is a diagram of the unknot. Call such a diagram U -diagram. We start to trace our diagram from X of the first long horizontal strand and count the number of crossing changes needed to turn it into U -diagram. We call the crossing that we don't need to change "right crossings", and these we have to change "wrong crossings". Let us cut long horizontal strands at their left ends marked X . We obtain t strings. Numerate them from 1 to t in the order we trace the diagram. For example, the first long horizontal strand belongs to the t -th string. In the word for this knot the i -th string corresponds to the part $x^{\alpha_i}y^{\beta_i}$. Each string goes around in A -part without self intersections, then goes down the vertical strand, goes around in B -part, probably intersecting itself on the long vertical strand then returns to A -part via the long horizontal strand where it may intersect itself. It shows that self intersections of each string are right crossings, because we trace a vertical segment of a string before tracing the corresponding horizontal segment. The number of self intersections on the long vertical strand of the i -th string is β_i , so the total number of self intersections on $B \cup C$ is $\sum_1^t \beta_i = b$.

Next we analyze intersections between i -th and j -th strings inside $B \cup C$. Note that a crossing (i, j) with vertical segment belonging to the i -th string and the horizontal segment belonging to the j -th string is right if $i < j$ and is wrong if $i > j$.

There are two possible positions of these strings in C , corresponding to the two possible scenario drawn on Figure 7a,b.

We introduce $\varepsilon_{ij} = 1, -1$ or 0 according to the rule:

$$\varepsilon_{ij} = \begin{cases} 1, & \text{if in the upper left corner of the rectangle formed by long vertical} \\ & \text{and long horizontal strands of } i\text{-th and } j\text{-th strings, the vertical} \\ & \text{string is greater than the horizontal;} \\ -1, & \text{if the vertical string on that corner is less than the horizontal;} \\ 0, & \text{if they are equal.} \end{cases}$$

Let c_{ij} be the number of intersections between the strings i, j in part $B \cup C$ (self intersections are not included). Then the number of wrong intersections among them is $\frac{1}{2}(c_{ij} + \varepsilon_{ij})$. By $\sum_{(i,j)}$ we mean the sum over unordered pairs (i, j) $1 \leq i, j \leq t$, $i \neq j$. Denote $N_B = \sum_{(i,j)} \varepsilon_{ij}$. Summing up over all pairs of strings we get

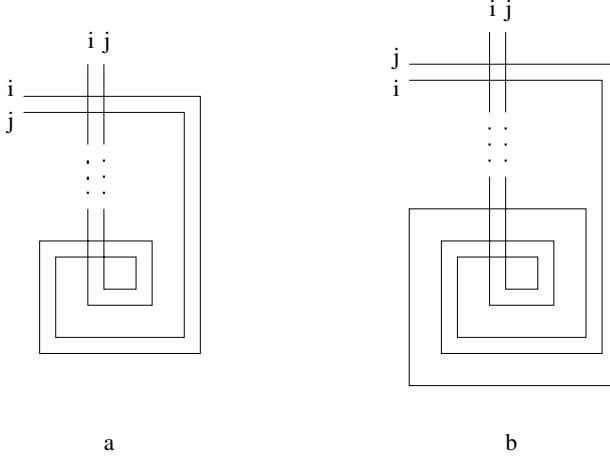


Figure 7

the number of wrong crossings in part $B \cup C$ to be $u_{B \cup C} = \frac{1}{2}(c_{B \cup C} - b + N_B) = \frac{1}{2}(b(t-1) - \sum_1^t \nu_k + \frac{t(t+1)}{2} + N_B)$

Calculation of the number of wrong intersections in part $A \cup C$, $u_{A \cup C}$, is similar. We cut each long horizontal strand at its right end O , so that the first long horizontal strand that belonged to the t -th string now belongs to the first string, and k -th long horizontal strand if belonged to $(i-1)$ -th string now belongs to i -th string. Denote the central square with the new numeration by C_A , and with the old numeration by C_B . Now in each string long horizontal strand goes before long vertical strand. We can calculate the number of intersections (i, j) where horizontal segment j is greater than vertical segment i , using the same method we used in B -part (for this calculation horizontal strands play the role of vertical strands in B). Let

$$\delta_{ij} = \begin{cases} 1, & \text{if in the lower right corner of the rectangle formed} \\ & \text{by } i\text{-th and } j\text{-th strings in } C_A, \text{ the horizontal} \\ & \text{string is greater than the vertical string,} \\ -1, & \text{if the horizontal string is smaller than the vertical one,} \\ 0, & \text{if they are equal.} \end{cases}$$

Let $N_A = \sum_{(i,j)} \delta_{ij}$. The number of intersections (i, j) in $A \cup C$ with $j > i$ is

$$u_{j>i} = \frac{1}{2}[c_{A \cup C} - a + N_A].$$

These are right intersections. Also all a intersections for $j = 1$ are right. All the rest intersections: with $i > j$ and $i = j \neq 1$ are wrong, and there are $u_{A \cup C}$ of them:

$$\begin{aligned} u_{A \cup C} &= c_{A \cup C} - \frac{1}{2}(c_{A \cup C} - a + N_A) - a = \frac{1}{2}(c_{A \cup C} - a - N_A) \\ &= \frac{1}{2}(a(t-1) - \sum_1^t \mu_k + \frac{t(t+1)}{2} - N_A) \end{aligned}$$

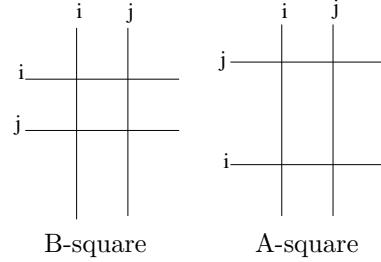
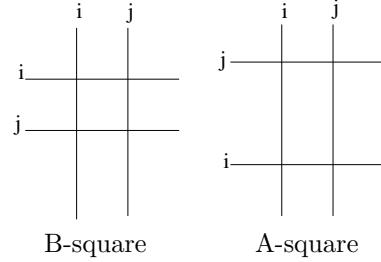
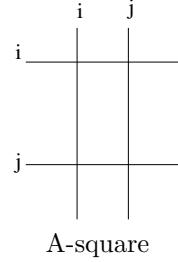
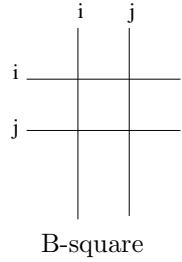


Figure 8a.

Figure 8b.

The number of wrong intersections in C is clearly $\frac{t(t-1)}{2}$ so the total number of wrong intersections on the diagram is $U = u_{A \cup C} + u_{B \cup C} - \frac{t(t-1)}{2}$, so

$$u(k) \leq U = \frac{1}{2}((a+b)(t-1) - \sum_1^t \mu_k - \sum_1^t \nu_k + 2t + N_B - N_A).$$

To prove Theorem(6.1) it remains to prove that $N_B - N_A = -(t-1)$.

The proof of this statement uses only some combinatorics on the square and is not related to a knot.

Suppose we have two squares A -square and B -square with t vertical strands, numerated from 1 to t , and t horizontal strands numerated from 1 to t , so that in B -square the uppermost horizontal strand has number t . In A -square the numeration of vertical strands coincides with that of B -square, the uppermost horizontal strand changes number from t to 1, and all other horizontal strands if had number i , get number $i+1$. (A - and B -squares inherit numeration of the strands from the cuts of the long horizontal strands at O's and at X's respectively).

Lemma 7.2. *When we interchange vertical strands, $N_B - N_A$ does not change.*

Proof. In the picture we see the fragments of B and A squares with rectangles formed by i -th and j -th strands. These two configurations describe all possible mutual positions of i -th and j -th strands up to interchanging i -th and j -th vertical strands, which is an involution. So it is enough to prove the statement for these two configurations. When we interchange vertical strands on Figure 8a, ε_{ij} and δ_{ij} change from 0 to 1 if $i > j$ and from 0 to -1 if $i < j$. So $\varepsilon_{ij} - \delta_{ij}$ does not change. If the position of i -th and j -th strands is as on Figure 8b then ε_{ij} changes from 0 to -1 , δ_{ij} changes from 1 to 0 if $i > j$, and ε_{ij} changes from 0 to 1, δ_{ij} changes from -1 to 0, if $i < j$ and $\varepsilon_{ij} - \delta_{ij}$ again does not change. So $N_B - N_A$ does not change when we change the order of vertical strands. \square

Now we prove that $N_B - N_A = -(t-1)$. Change the order of vertical strands so that they go from 1 to t . Then there is one-to-one correspondence $(i, j) \leftrightarrow (i+1, j+1)$ between the intersection points of B -square (i, j) , $i, j \neq t$ and intersection points of A -square with $i, j \neq 1$. We have $\varepsilon_{ij} - \delta_{i+1, j+1} = 0$. So

$$\sum_{i,j \neq t} \varepsilon_{ij} - \sum_{i,j \neq 1} \delta_{ij} = 0.$$

(Summation is over non-ordered pairs (i, j) .) Now $\varepsilon_{t,j} = -1$ for $j = 1, 2, \dots, t-1$ and $\delta_{1,j} = 0$ for $j = 2, \dots, t$. So $N_B - N_A = -(t-1)$. This completes the proof of Theorem 6.1. \square

We have proven

$$u(K) \leq \frac{1}{2}((a+b)(t-1) - \sum_1^t \mu_i - \sum_1^t \nu_i + (t+1)) = A(x^-).$$

Combining this with inequalities

$$A(x^-) \leq -\tau(K), \quad |\tau(K)| \leq u(K),$$

we obtain the unknotting number for Lorenz knots:

$$u(K) = \frac{1}{2}((a+b)(t-1) - \sum_1^t \mu_i - \sum_1^t \nu_i + (t+1)).$$

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